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Evolution of the gluon density in x with a running coupling constant

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Abstract. We describe the calculation of the evolution of the gluon density in x from an initial value $x = x_0 = 0.01$ to smaller values, up to $x = 10^{-8}$ in the hard pomeron formalism with a running coupling introduced on the basis of the bootstrap equation. The obtained gluon density is used to calculate the singlet part of the proton structure function. Comparison with experiment and the results following from the fixed coupling evolution is made.

1 Introduction

Recent results obtained at HERA [1,2] may be interpreted as a manifestation of the hard pomeron, which naturally explains a sharp rise of $F_2(x, Q^2)$ at low x. The original BFKL hard pomeron, however, has a drawback of treating the coupling constant as fixed, since it sums only powers of $\log 1/x$ and not those of $\log Q^2$. A rigorous way to introduce a running coupling into it still remains beyond the possibilities of the theory, since it inevitably involves a problem of low Q^2 behaviour and thus of confinement. In a series of papers [3–6] we have adopted a more intuitive way to attack this problem, based on the so-called bootstrap relation [7,8], which is, in fact, the unitarity condition for the t-channel with a colour quantum number of a gluon. It ensures that the one-reggeized-gluon exchange supposed to give a dominant contribution in this channel is unitary by itself, which is a necessary requirement to use it as an input for the construction of the BFKL pomeron.

Assuming that this fundamental requirement should be preseved in the theory with a running coupling, we proposed a minimal modification of the BFKL pomeron equation which, on the one hand, satisfies the bootstrap condition and, on the other hand, leads to the standard results with the running coupling in the double log (DL) limit, i.e. when leading terms in the product $\log 1/x \log Q^2$ are summed. This modification reduces to the substitution of every momentum squared k^2 in the pomeron equation by a function $\eta(k)$ which at large k behaves as $k^2/2\alpha_s(k^2)$. The behaviour of $\eta(k)$ at low k remains beyond any theoretical controle. We parametrize it as

$$\eta(k) = (b_0/8\pi)(k^2 + m^2) \ln \frac{k^2 + m^2}{\Lambda^2} \qquad (1)$$

where Λ is the standard QCD parameter, $b_0 = 11 - (2/3)N_f$ and the effective gluon mass m simulates both the confinement effects and the freezing of the coupling.

Solving numerically the pomeron equation in this approach we found two supercritical pomerons [5]. Adjusting the mass m to fit the experimental slope of the leading pomeron of 0.25 $(\mathrm{GeV/c})^{-2}$ we obtained for their intercepts

$$\Delta_0 = 0.384, \quad \Delta_1 = 0.191$$

and the slope of the subdominant pomeron results $\alpha_1' = 0.124$ $(\mathrm{GeV/c})^-2$. Calculating observable quantities with only these two asymptotic states taken into account we found that the picture which emerges, in all probability, corresponds to energies much higher than the present ones [5]. In particular the average $\langle k_\perp \rangle$ was found to be very large (~ 10 GeV/c) and independent of energy, which may indicate a saturation of its growth observed at present energies [6].

To describe the present experimental data it is then necessary to take into account all the states from the pomeron equation spectrum. This can be achieved by converting the pomeron equation into an evolution equation in $1/x$ and solving it with an initial condition at some (presumably small) value $x = x_0$. In such an approach, taking a nonperturbative input at $x = x_0$ adjusted to the experimental data, also the problem of coupling the pomeron to the hadronic target is solved in an effective way.

This note is devoted to realizing such a program. In Sect. 2 we state our basic equations. The most difficult part of the program is to pass from the gluon density to the observable structure function. It is discussed in Sect. 3. Section 4 is devoted to fixing the initial gluon distribution for the future evolution. In Sect. 4 we present our numerical results. Section 5 contains a discussion and some conclusions.

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2 Basic equations

For the forward scattering amplitude the pomeron equation reads

$$(H - E)\psi = \psi_0 \tag{2}$$

Here ψ is a semi-amputated (one leg amputated only) pomeron wave function; $E = 1 - j$ is the pomeron "energy", related to its complex angular momentun j; $H = T + V$ is the "Hamiltonian" consisting of the kinetic energy given by the sum of the two gluon Regge trajectories, $T = -2\omega$ and of the potential energy V. With a running coupling introduced according to [3,4] both are expressed via the mentioned function η (1);

$$T(k) = \frac{N_c}{(2\pi)^2} \int \frac{d^2k' \eta(k)}{\eta(k')\eta(k-k')} \tag{3}$$

and

$$V\psi(k) = -\frac{T_1 T_2}{(2\pi)^2} \int d^2k' \psi(k') \times \left(\frac{2}{\eta(k-k')} - \frac{\eta(0)}{\eta(k)\eta(k')} \right) \tag{4}$$

where N_c is the number of colours and $T_{1(2)}$ are the colour operators for the two interacting gluons; in the vacuum channel we have $T_1 T_2 = -N_c$. Finally, the inhomogeneous term ψ_0 represents the interaction vertex between the pomeron and the hadronic target.

Taking the Mellin transformation of (2) one converts it into an evolution equation in $1/x$:

$$\frac{\partial}{\partial \ln 1/x} \psi(x,k) = -H\psi(x,k) \tag{5}$$

which should be supplemented with an initial condition at some $x = x_0$

$$\psi(x_0, k) = \psi_0(k) \tag{6}$$

containing the nonperturbative input about the coupling to the hadronic target.

The physical interpretation of the pomeron wave function is provided by the fact that in the DL approximation (5) reduces to an equation for the fully amputated function $\phi(x,k) = \eta(k)\psi(x,k)$:

$$\frac{\partial}{\partial \ln k^2} \frac{\partial}{\partial \ln 1/x} \phi(x,k) = \frac{3\alpha_s(k^2)}{\pi} \phi(x,k) \tag{7}$$

which coincides with the standard equation for the unintegrated gluon density $xg(x,k^2)$ in the DL limit. In fact, this circumstance lies at the root of our method to introduce a running coupling into the scheme. Thus we may identify

$$\phi(x,k) = cxg(x,k) \tag{8}$$

The normalizing factor c cannot be determined from the asymptotic (7). We shall be able to fix it by studying the coupling of the pomeron to the incoming virtual photon in the next section.

3 Coupling to the virtual photon

Once the function ϕ proportional to the gluon density is determined, one has to couple it to the projectile particle to calculate observable quantities. In particular, to find the structure function of the target one has to couple the gluons to the incoming virtual photon, that is, to find the colour density $\rho(q,k)$ which connects the photon of momentum q to the gluon of momentum k. This problem is trivial within the BFKL approach with a fixed small coupling. Then it is sufficient to take the colour density in the lowest order, which corresponds to taking for it the contribution of a pure quark loop into which the incoming photon goes $\rho_0(q,k)$.

The problem complicates enormously when one tries to introduce a running coupling into ρ. Then one has to take into account all additional gluon and $q\bar{q}$ pair emissions which supply powers of the logarithms of transverse momenta. Apart from making the coupling run, they will evidently change the form of $\rho(q,k)$. Unfortunately the bootstrap relation can tell us nothing about the ultimate form of the colour density with a running coupling, which essentially belongs to the t-channel with a vacuum colour quantum number. So we have to find a different way to introduce a running coupling into ρ.

A possible systematic way to do this consists in applying to the photon-gluon coupling the DGLAP evolution equation. One may separate the colour density from the rest of the amplitude by restricting its rapidity range to some maximal rapidity $y_0 \sim \log Q^2$ (which, of course should be much smaller than the overall rapidity $Y \sim \log Q^2/x$). Then the kinematical region of $\rho(q,k)$ will admit the standard DGLAP evolution in Q^2. Solving this equation one will find the quark density at scale Q^2 of the gluon with momentum k (i.e. essentially the structure function of the gluon with the virtuality k^2). This is exactly the quantity needed to transform the calculated gluon density created by the target into the observable structure function of the target. As a starting point for the evolution one may take the perturbative colour density ρ_0 at some low Q^2 when the logs of the transverse momenta might be thought to be unimportant.

This ambitious program, combining both evolution in both $1/x$ and Q^2, does not, however, look very simple to realize. As a first step, to clearly see the effects of the introduction of a running coupling according to [3, 4], we adopt a more phenomenological approach here, trying to guess a possible correct form for $\rho(q,k)$ on the basis of simple physical reasoning and also using the DL approximation to fix its final form.

With a pure perturbative photon colour density one would obtain for the $\gamma^* p$ cross-section

$$\sigma(x,Q^2) = \int \frac{d^2k \rho_0(q,k)\phi(x,k)}{(2\pi)^2 \eta^2(k)} \tag{9}$$

In fact, the projectile particle should be coupled to the full pomeronic wave function ϕ/η^2. From the physical point of view this expression is fully satisfactory for physical particles. However it is not for a highly virtual projectile.

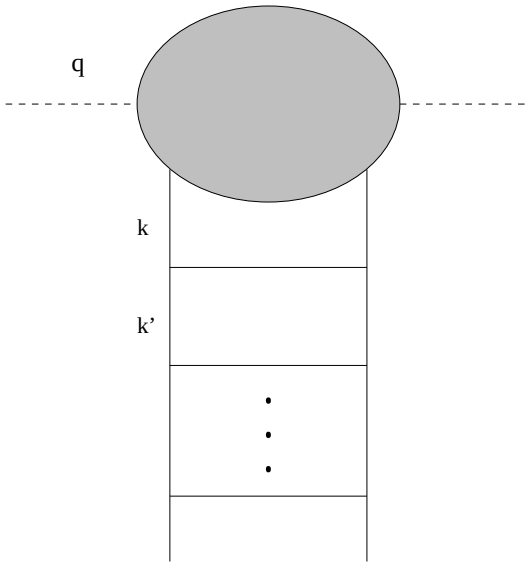

Fig. 1. The forward amplitude for a pomeron coupled to a virtual photon

To see this, we first note that for the forward amplitude our method of introducing a running coupling reduces to a very simple rule: the scale at which the coupling should be taken is given by the momentum of the emitted real gluon ($(k-k')^2$ in the upper rung in Fig. 1). Now take Q^2 very large and apply the DL approximation. Then the momenta in the ladder become ordered from top to bottom

$$Q^2 >> k^2 >> k'^2 >> .....$$

In this configuration, as can be traced from (2) and (9), all α_s's acquire the right scale (i.e. corresponding to the DGLAP equation) except for the upper rung: $\alpha_s(k^2)$ appears twice. This defect can be understood if one notices that the upper gluon is, in fact, coupled to a virtual particle. If this particle were a gluon, then the interaction (4) would cancel one of the two $\alpha(k^2)$'s and substitute it by an α taken at the scale corresponding to its own virtuality. We assume that something similar should take place also for virtual quarks to which the gluon chain may couple. The scale of the particle momenta squared which enter the upper blob in Fig. 1 should have the order Q^2 (this is the only scale that remains after these momenta are integrated out). As a result the lowest order density should be rescaled according to

$$\rho_0(q,k) \to \frac{\alpha(Q_1^2)}{\alpha(k^2)}\rho_0(q,k) \tag{10}$$

where Q_1^2 has the same order as Q^2.

The approximation we assume in this paper is that the substitution (10) is sufficient to correctly represent the photon colour density with a running coupling. We shall check its validity by studying the quark density which results from (10) in the DL approximation and comparing it with the known result based on the DGLAP equation.

Explicitly the zeroth order density ρ_0 has the following forms for the transverse (T) and longitudinal (L) photons (see e.g. [9] and do the integration in the quark loop momenta)

$$\rho_0^{(T)}(q,k) = \frac{3e^2}{8\pi^2}\sum_f Z_f^2 \int_0^1 d\alpha \Big((\alpha^2+(1-\alpha)^2)((1+2z^2) \times g(z) - 1) + \frac{\zeta}{\alpha(1-\alpha)+\zeta}(1-g(z)) \Big) \tag{11}$$

$$\rho_0^{(L)}(q,k) = \frac{3e^2}{2\pi^2}\sum_f Z_f^2 \int_0^1 d\alpha \frac{(\alpha(1-\alpha))^2}{\alpha(1-\alpha)+\zeta} \times (1-g(z)) \tag{12}$$

Here the summation goes over the quark flavours. The dimensionless variables ζ and z are defined as

$$\zeta = \frac{m_f^2}{Q^2}, \quad z = \frac{k^2}{4Q^2}\frac{1}{\alpha(1-\alpha)+\zeta} \tag{13}$$

and m_f and Z_f are the mass and charge of the quark of flavour f. The function $g(z)$ is given by

$$g(z) = \frac{1}{2z\sqrt{z^2+1}} \ln \frac{\sqrt{z^2+1}+z}{\sqrt{z^2+1}-z} \tag{14}$$

The structure function is obtained from the cross-section by the standard relation

$$F_2(x,Q^2) = \frac{Q^2}{\pi e^2}(\sigma^{(T)}+\sigma^{(L)}) \tag{15}$$

In the DL limit only the transverse cross-section contributes. We can also neglect the quark masses in this approximation. Then, with a substitution (10), from (9), (11) and (15) we obtain an expression for the quark (sea) density of the target

$$xq(x) = \frac{3}{\pi^2 b_0^2}\frac{Q^2}{\ln Q^2}\int^{Q^2}\frac{dk^2\phi(x,k)}{k^4\ln k^2}\int_0^1 d\alpha(\alpha^2+(1-\alpha)^2) \times((1+2z^2)g(z)-1) \tag{16}$$

where $g(z)$ is given by (14) and we assumed that large values of $k^2 < Q^2$ contribute in accordance with the DL approximation. In this approximation the asymptotics of the gluon density $xg(x,k^2)$ and consequently of $\phi(x,k^2)$ is known:

$$\phi(x,k^2) = cxg(x,k^2) \simeq c\exp\sqrt{a\ln\frac{1}{x}\ln\ln k^2} \tag{17}$$

where $a = 48/b_0$. Putting (17) into (16), after simple calculations we find the asymptotical expression for the quark density

$$xq(x,k^2) \simeq \frac{4c}{\pi^2 b_0^2}\sqrt{\frac{\ln\ln k^2}{a\ln 1/x}}\exp\sqrt{a\ln\frac{1}{x}\ln\ln k^2} \tag{18}$$

On the other hand, from the DGLAP equation we find, with the same normalization

$$xq(x,k^2) \simeq \frac{4}{3b_0^2}\sqrt{\frac{\ln\ln k^2}{a\ln 1/x}}\exp\sqrt{a\ln\frac{1}{x}\ln\ln k^2} \tag{19}$$

As we observe the approximation (10) for the colour density of the photon projectile leads to the correct relation between the quark and gluon densities in the DL limit. This justifies the use of (10), at least for high $1/x$ and Q^2. Comparing (18) and (19) we also obtain the normalization factor c which relates the pomeron wave function to the gluon density

$$c = \pi^2 b_0/3 \tag{20}$$

4 The initial distribution

To start the evolution in $1/x$ we have to fix the initial gluon density at some small value $x = x_0$. Evidently, the smaller is x_0, the smaller is the region where we can compare our predictions with the experimental data. On the other hand, if x_0 is not small enough, application of the asymptotic hard pomeron theory becomes questionable. Guided by these considerations we choose $x_0 = 0.01$ as our basic initial x although we also tried $x = 0.001$ to see the influence of possible subasymptotic effects.

The initial wave function $\phi(x_0, k^2)$ has to be chosen in accordance with the existing data at $x = x_0$ and all k^2 available. The experimental F_2 is a sum of the singlet and nonsinglet parts, the latter giving a relatively small contribution at $x = 0.01$. Our theory can give predictions only for the singlet part (and one of the criteria for its applicability is precisely the relative smallness of the nonsinglet contribution). The existing experimental data at $x = 0.01$ give values for F_2 averaged over rather large intervals of x and Q^2. For all these reasons, rather than to try to adjust our initial $\phi(x_0, k^2)$ to the pure experimental data, we have preferred to match it with the theoretical predictions for the gluon density and the singlet part of F_2 given by some standard parametrization fitted to the observed F_2 in a wide interval of Q^2 and small x. As such we have taken the GRV LO parametrization [10]. The choice of LO has been dictated by its comparative simplicity and the fact that at $x = 0.01$ the difference betwen LO and NLO is insignificant.

Thus, for the initial distribution we have taken the GRV LO gluon density with an appropriate scaling factor. Putting this density into (8), (9) and (15) one should be able to reproduce the sea quark density and thus the singlet part of the structure function. In the GRV scheme the relation between the gluon density and the quark density is much more complicated and realized through the DGLAP evolution. Since the DGLAP evolution and the pomeron theory are not identical, one should not expect that our initial gluon density should exactly coincide with the GRV one to give the same singlet structure funcction. One has also to have in mind the approximate character of our colour density ρ at small Q^2. In fact, with the initial ϕ given by (8) and the gluon density exactly taken from the GRV parametrization at $x = 0.01$ we obtain a 30% smaller values for the singlet part of the structure function as given by the same GRV parametrization, the difference growing at low Q^2. To make the description better we used a certain arbitrariness in the scale Q_1^2 which enters (10) and also the scale at which the coupling freezes in the density ρ. The optimal choice to fit the low Q^2 data is to take

$$\alpha(Q_1^2) = \frac{4\pi}{b_0} \frac{1}{\ln((0.17 * Q^2 + 0.055\ (\mathrm{GeV/c})^2)/\Lambda^2)} \tag{21}$$

With this $\alpha(Q_1^2)$ the obtained singlet structure function at $x = 0.01$ has practically the same Q^2 dependence as the GRV one, although it results 30% smaller in magnitude. This mismatch can be interpreted in two different ways. Either we may believe that the gluon density given by the GRV is the correct one and the deficiency in the singlet part of the structure function is caused by our approximate form of the colour density ρ (which is most probable). Or we may think that the colour density to be used in the DGLAP should coincide with ours only for large enough Q^2 and $1/x$ and at finite values they may somewhat differ (our relation (8) was established strictly speaking only in the DL limit). Correspondingly we may either take the relation (8) as it stands and use the GRV LO gluon density at $x = 0.01$ in it, or introduce a correcting scaling factor 1.3 which brings the structure function calculated with the help of (9)–(15) into agreement with the GRV predictions. In the following we adopt the second alternative, that is we assume that our initial gluon distribution at $x = 0.01$ is 30% higher that the one given by the GRV parametrization. The singlet part of the structure function at $x = 0.01$ calculated from (9)–(15) with this choice is shown in Fig. 2 together with the GRV predictions. However one can easily pass to the first alternative by simply reducing our results by factor 1.3.

5 Evolution: numerical results

With the initial wave function $\phi(x = 0.01, k^2)$ chosen as indicated in the preceding section we solved the evolution equation for $10^{-8} < x < 10^{-2}$. The adopted calculational scheme was to diagonalize the Hamiltonian in (2), reduced to one dimension in the transverse momentum space after angular averaging, and represent the initial wave function as a superposition of its eigenvectors. To discretize k^2 a grid was introduced, after which the problem is reduced to a standard matrix one. To check the validity of the obtained results we have also repeated the evolution using a Runge-Kutta method, resulting in a very good agreement. The final results obtained for the gluon distribution $xg(x, Q^2)$ as a function of x for various Q^2 are shown in Fig. 3 where the first two plots correspond to x and Q^2 presently available, whereas the last two plots show the behaviour of the calculated gluon density in the region up to very small x and very high Q^2, well beyond the present possibilities. For comparison we have also shown the gluon densities for the GRV LO parametrization [10], for the MRS parametrization [11] and also for the pure BFKL evolution as calculated in [12].

Putting the found gluon densities into (9)–(15) we obtain the (singlet part of) proton structure function $F_2(x, Q^2)$. The results are illustrated in Fig. 4 for the x dependence. As for the gluon densities, the experimentally investigated region is shown in the first two plots,

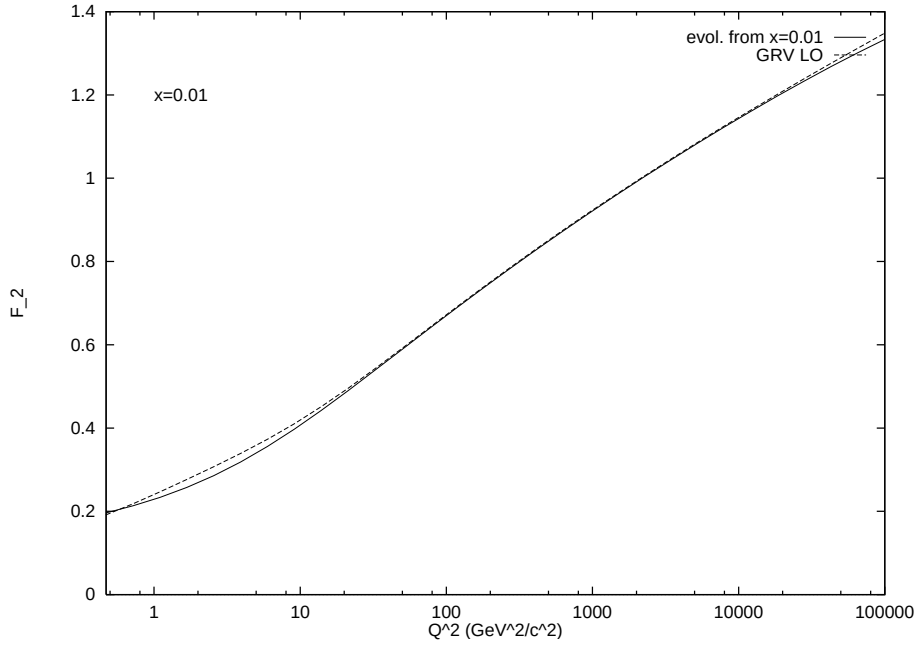

Fig. 2. The singlet part of the structure function of the proton at $x = 0.01$. The *continuous line* is the result of our calculation while the *dashed line* correspond to the GRV prediction

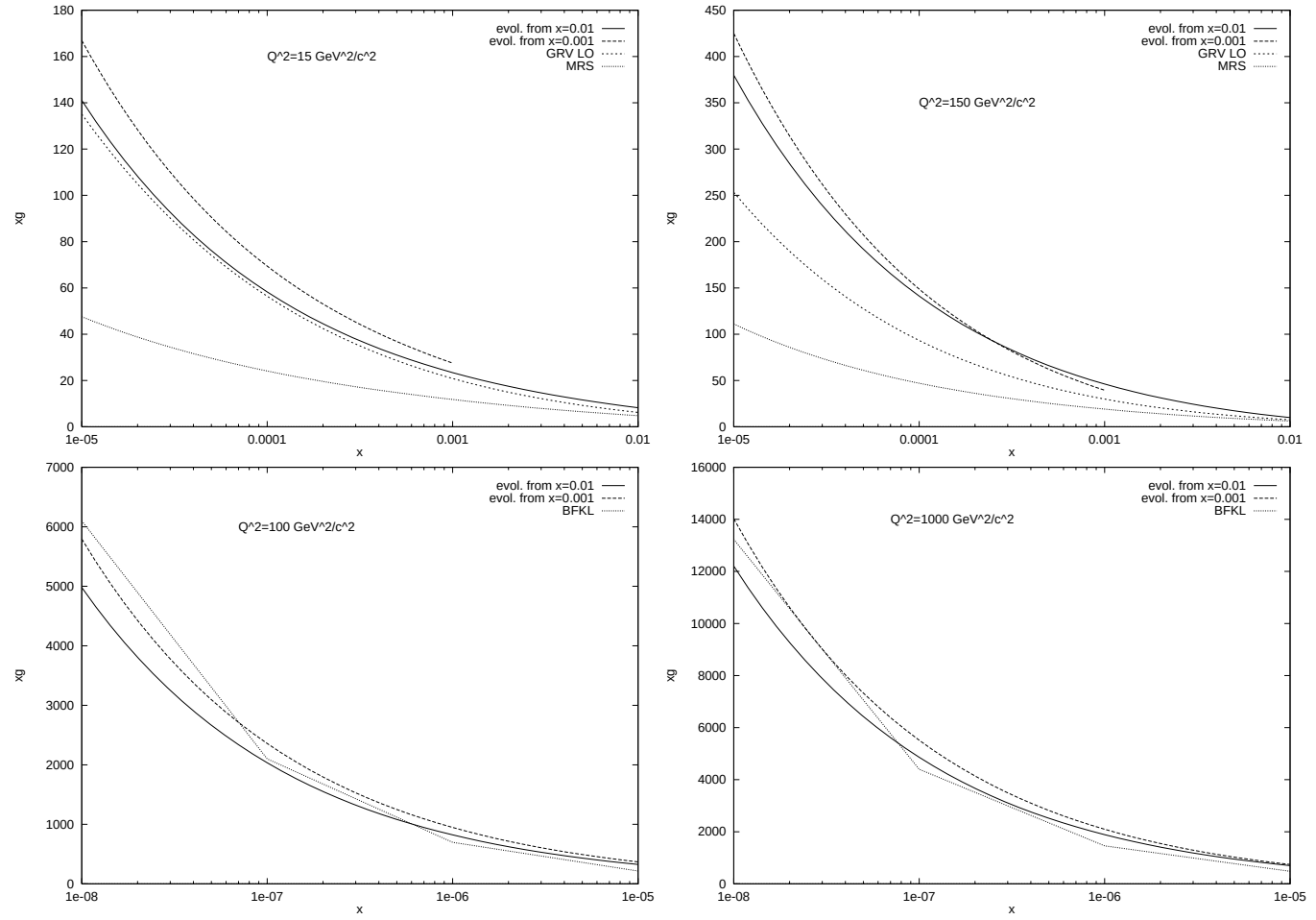

Fig. 3. The gluon distributions as a function of x evolved from $x = 0.01$ and $x = 0.001$ for the experimentally accessible kinematical region and for asymptotically high values of Q^2 and $1/x$. Standard DGLAP evolved parametrizations (GRV-LO and MRS) and the BFKL evolved distributions from [12] (we report only few points connected by lines) are shown for comparison

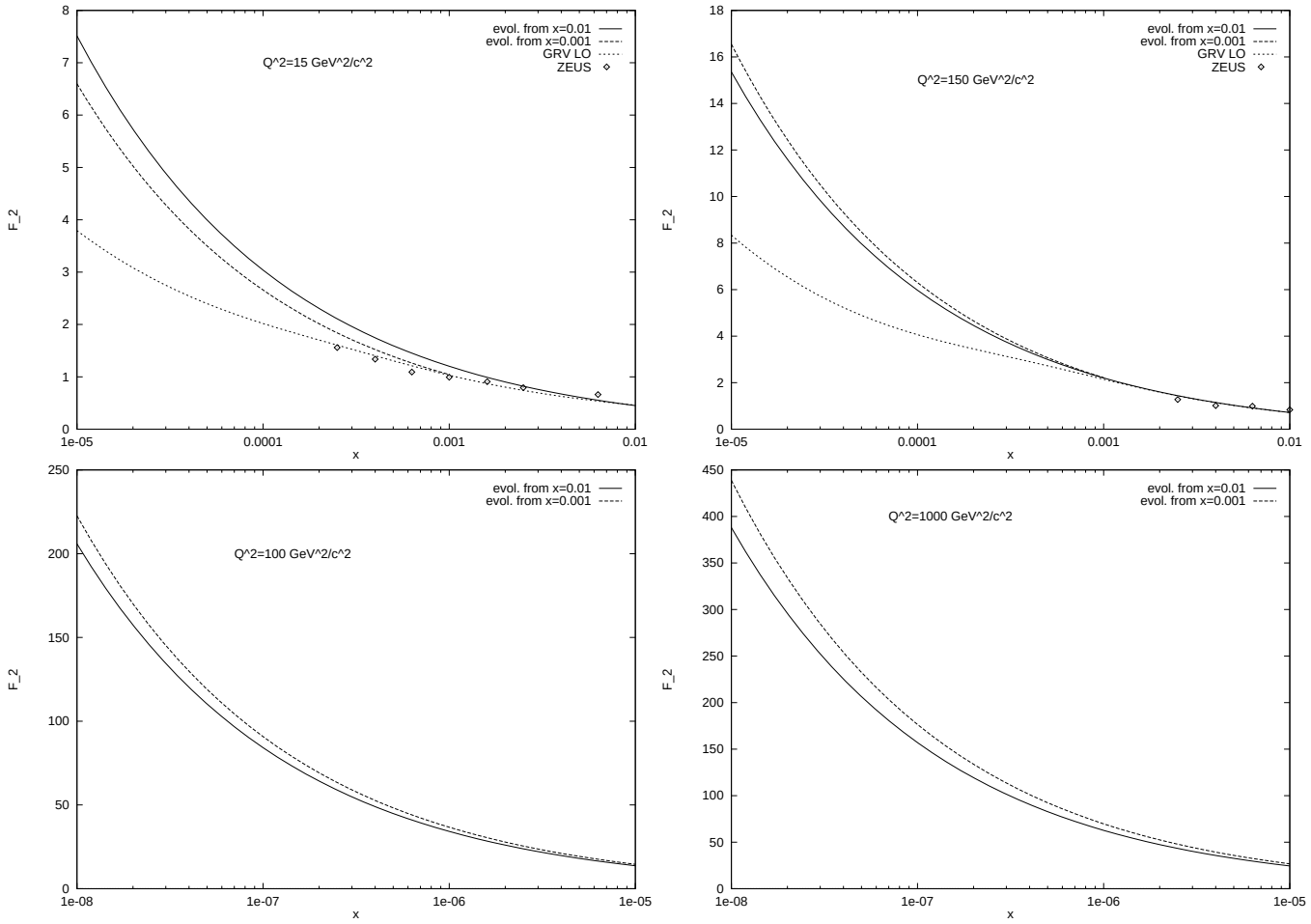

Fig. 4. x dependence of the singlet part of the proton structure function obtained by evolution from $x = 0.01$ and $x = 0.001$, compared to the GRV prediction and the ZEUS 94 data. Values for asymptotically high values of Q^2 and $1/x$ are also shown

in which the existing experimental data from [2] are also presented.

Finally, to see a possible influence of subasymptotic effects, we have repeated the procedure taking as a starting point for the evolution a lower value $x = 0.001$. The resulting gluon distributions and structure functions are also presented in the above figures.

6 Discussion and conclusions

To discuss the obtained results we have to remember that they involve two quantities of a different theoretical status. One is the pomeron wave function ϕ which can be identified with the gluon distribution (up to a factor) on a rather solid theoretical basis. The other is the quark density (which is equivalent to the structure function), for which we actually have no rule for the introduction of a running constant and which in the present calculation involves a semi-phenomenological ansatz (10). Evidently the results for the latter are much less informative as to the effect of the running coupling introduced in our way. Therefore we have to separately discuss our prediction for the gluon distribution, on the one hand, and for the structure function, on the other.

Let us begin with the gluon distribution. Comparing our results with those of GRV, which correspond to the standard DGLAP evolution, we observe that at high enough Q^2 and low enough x our distributions rise with Q and $1/x$ faster than those of GRV. This difference is, of course, to be expected. The hard pomeron theory in any version predicts a power rise of the distribution with $1/x$ to be compared with (19) for the DGLAP evolution. As to the Q-dependence, the fixed coupling (BFKL) hard pomeron model predicts a linear rise, again much stronger than (19). Our running coupling model supposedly leads to a somewhat weaker rise. From our results it follows that it is still much stronger than for the DGLAP evolution. However one can observe that these features of our evolution become clearly visible only at quite high Q and $1/x$. For moderate $Q < 10\,\mathrm{GeV/c}$ and/or $x > 10^{-4}$ the difference between our distributions and those of GRV is insignificant. As to the DGLAP evolved MRS parametrization, it gives the gluon distribution which lies systematically below the GRV one and, correspondingly, below our values, the difference growing with Q and $1/x$.

We can also compare our gluon distributions with the pure BFKL evolution (fixed coupling) results, as presented in [12]. One should note that the initial values for the evolution chosen in [12] are rather different from ours (borrowed from GRV). The initial gluon distribution in [12] is smaller than ours by a Q-dependent factor, equal to ~ 2.5 at $Q = 2$ GeV/c and ~ 1.4 at $Q = 30$ GeV/c. If one roughly takes that into account then one concludes that at moderate $1/x$ our evolution and the pure BFKL one lead to quite similar results. However at smaller x one observes that our running coupling evolution predicts a weaker rise with Q, as expected.

Passing to the structure functions we observe in Fig. 4 that our results give a somewhat too rapid growth with $1/x$ in the region $10^{-3} < x < 10^{-2}$ as compared to the experimental data (and also to the parametrizations GRV fitted to these data). With the scaling factor 1.3 introduced to fit the data at $x = 0.01$ we overshoot the data at $x < 10^{-3}$ by $\sim$25%. Without this factor we get a very good agreement for $x < 10^{-3}$ but are below experiment at $x = 0.01$ by the same order. This discrepancy may be attributed either to subasymptotic effects or to a poor quality of our ansatz (10). Comparison with the result obtained with a lower starting point for the evolution $x = 0.001$ shows that subasymptotic effects together with a correct form of coupling to quarks may be the final answer.

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